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THE EXISTENCE AND STABILITY OF STEADY STATES OF COMBUSTION IN A FLOW†

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A new approach which allows one to investigate combustion in flows for rather broad classes of multi-stage chemical reactions is employed. Combustion in flows for a single-state chemical reaction has been studied in a number of papers [1, 2]. There the process was described by a scalar parabolic equation and could be investigated by fairly simple mathematical techniques. When one considers combustion with more-complicated kinetics the methods that were used are not in general applicable.

1. STATEMENT OF THE PROBLEM

CONSIDER the parabolic system of equations

$$\frac{\partial u}{\partial t}=\mathbf{a}\frac{\partial^2 u}{\partial x^2}-c\frac{\partial u}{\partial x}-F(u) \tag{1.1}$$

$$u=(u_1,\ldots,u_n),\quad F(u)=(F_1(u),\ldots,F_n(u)),\quad \mathbf{a}=\alpha E,\quad \alpha>0$$

on the semi-axis $x\geqslant 0$ for $t\geqslant 0$. Here $F(u)$ is a sufficiently smooth vector function, $\mathbf{a}$ is a scalar matrix, E is the unit matrix and c is a positive constant (the flow velocity). The boundary conditions are specified in the form

$$u(0,t)=u^0,\quad u(+\infty,t)=0, \tag{1.2}$$

where u^0 is some constant vector.

It is assumed that the function $F(u)$ satisfies the following conditions:

1. $F(0)=0$ and $u=0$ is an asymptotically stable stationary point of the kinetic system

$$\partial u/\partial t=-F(u) \tag{1.3}$$

2. For $u\geqslant 0$ (a coordinatewise inequality) there exists a convex domain Π such that $u=u^0$ belongs to this domain, $u=0$ lies on its boundary, the functions $F_i(u)$ $(i=1,\ldots,n)$ are all positive inside this domain, and the domain Π is positively invariant with respect to system (1.3). The latter means that if the point u^0 belongs to the closure of the domain Π, then the solution $u(t)$ of Eq. (1.3) with initial condition $u(0)=u^0$ also belongs to Π.

3. The inequalities

$$\partial F_i/\partial u_j\geqslant 0,\quad i,j=1,\ldots,n,\quad i\neq j \tag{1.4}$$

are satisfied in the domain Π.

The first two conditions are typical for the equations of chemical kinetics with irreversible reactions [3, 4] for appropriate changes of variables. They mean that in the balance-polyhedron Π a stable equilibrium exists for the kinetic system, the rates of the reactions $F_i(u)$ being positive for positive concentrations, and that the concentrations of the reactants cannot become negative during the reaction.

Inequalities (1.4) are satisfied by wide classes of reactions [3–6], which is important later, because in this case comparison theorems hold for Eq. (1.1): if $f_1(x)\leqslant f_2(x)$ for $x\geqslant 0$ and $f_i(0)=u^0$, $i=1, 2$, then the solutions

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$u_i(x, t)$ of Eq. (1.4) with initial conditions $u_i(x, 0) = f_i(x)$ satisfy the inequalities $u_1(x, t) \leqslant u_2(x, t)$ for $x \geqslant 0$ and $t \geqslant 0$.

2. EXISTENCE AND STABILITY OF STEADY-STATE SOLUTIONS

We shall show that problem (1.1), (1.2) has a steady-state solution. To do this we set the initial condition in the form

$$u(x, 0) = u^0, \quad x \geqslant 0 \tag{2.1}$$

and consider problem (1.1), (1.2) (2.1). Its solution $u(x, t)$ is contained in Π for all $x \geqslant 0$, $t \geqslant 0$ [7] and by condition (1.3) does not increase monotonically in t for all fixed x. Hence as $t \to +\infty$, $u(x, t)$ tends to some function $\mathbf{w}(x)$. One can verify that this function satisfies Eq. (1.1) and boundary conditions (1.2).

Furthermore, it is obvious that $\mathbf{w}(x)$ belongs to Π and does not increase with x. The latter follows from the fact that if for some $x_0 \geqslant 0$ and i we have $\mathbf{w}_i^{\bullet}(x_0) > 0$, then

$$\mathbf{w}_i^{\bullet\bullet}(x_0) \geqslant c\alpha^{-1}\mathbf{w}_i^{\bullet}(x_0) > 0$$

and the function $\mathbf{w}_i$ will increase without limit as $x \to \infty$, when, by assumption, it cannot exceed the value u_i^0. Thus the limit of $\mathbf{w}(x)$ as $x \to \infty$ exists and is a stationary point of system (1.3).

We shall denote this stationary point by $\mathbf{w}^+$ and show that $\mathbf{w}^+ = 0$. We will assume the contrary. Then $0 \leqslant \mathbf{w}^+ \leqslant u^0$ and $\mathbf{w}^+ \neq u^0$. Because of the positive invariance of the domain Π and inequalities (1.4), the domain Π^+ consisting of points from the set Π satisfying the condition $u \geqslant \mathbf{w}^+$ is positively invariant relative to Eq. (1.3). On the other hand, the interval τu^0 joining the points $u = 0$ and $u = u^0$, lies in Π for $0 < \tau \leqslant 1$ and along this interval $F(u) > 0$. Hence the solution of Eq. (1.3) with initial condition $u(0) = \tau_0 u^0$, where τ_0 is such that $\tau_0 u^0$ lies on the boundary of the set Π^+, cannot remain inside this set, which contradicts its positive invariance. Thus we have shown that $\mathbf{w}(x) \to 0$ as $x \to \infty$.

The concept of functional undecomposibility of a matrix will be necessary later. The continuous matrix $B(x)$ is called functionally undecomposible if the matrix formed by the norms of the elements of the matrix $B(x)$ in the space C is undecomposible. We note that from the inequality $\mathbf{w}^{\bullet}(x) \leqslant 0$ for $x \geqslant 0$ and for functional undecomposibility of the matrix $F^{\bullet}[\mathbf{w}(x)]$ for $x \geqslant 0$ it follows† that $\mathbf{w}^{\bullet}(x) < 0$ for $x > 0$. The inequality $\mathbf{w}^{\bullet}(0) \leqslant 0$ follows from $\mathbf{w}^{\bullet\bullet}(0) \leqslant 0$ and $F(u^0) > 0$.

The function $\mathbf{w}^{\bullet}(x)$ is therefore negative for $x \geqslant 0$ and tends to zero as $x \to \infty$. From this and the assumption of functional undecomposibility of the matrix $F^{\bullet}[\mathbf{w}(x)]$ it follows that the equation

$$\begin{gathered} a u^{\bullet\bullet} - c u^{\bullet} - F^{\bullet}(\mathbf{w}(x)) u = \lambda u \\ u(0) = 0, \quad u(+\infty) = 0 \end{gathered} \tag{2.2}$$

has no non-zero solutions for $\operatorname{Re}\lambda \geqslant 0$ [8]. This means that the steady-state solution $\mathbf{w}(x)$ of problem (1.1), (1.2) is stable with respect to small perturbations in the space of continuous functions with conditions (2.2), i.e. there exists an $\varepsilon > 0$ such that the inequality

$$\sup_{x \geqslant 0} |f(x) - \mathbf{w}(x)| \leqslant \varepsilon$$

implies convergence:

$$\sup_{x \geqslant 0} |u(x, t) - \mathbf{w}(x)| \to 0 \quad \text{as} \quad t \to \infty$$

where $u(x, t)$ is the solution of problem (1.1), (1.2) with initial condition

$$u(x, 0) = f(x), \quad (f(0) = u^0, \quad f(+\infty) = 0) \tag{2.3}$$

We shall now show that the solution $\mathbf{w}(x)$ is stable not just with respect to small perturbations, but also globally. We first assume that $f(x)$ does not increase when $x \geqslant 0$. It follows from the comparison theorem that it is sufficient to consider the cases $f(x) \geqslant \mathbf{w}(x)$ for $x \geqslant 0$ and $f(x) \leqslant \mathbf{w}(x)$ for $x \geqslant 0$. We shall only consider the first case here because they are similar.

† VOL'PERT A. I. and VOL'PERT V. A., Travelling waves described by monotonic parabolic systems. Chernogolovka Preprint, OIKhF Akad. Nauk SSSR, 1990.

Together with problem (1.1), (1.2) we shall consider n initial boundary-value problems for Eq. (1.1) in the domain $x \geq i\mathbf{h}$ ($i = 1, 2, \ldots, n$) where $\mathbf{h}$ is some positive number, with boundary conditions

$$u(i\mathbf{h}, t) = u^0, \quad u(+\infty, t) = 0$$

It is obvious that $\mathbf{w}_i(x) \equiv \mathbf{w}(x - i\mathbf{h})$ are steady-state solutions of these problems.

We choose $\mathbf{h}$ and n so that

$$\sup_{x \geq i\mathbf{h}} |\mathbf{w}_i(x) - \mathbf{w}_{i-1}(x)| \leq \varepsilon/2 \tag{2.4}$$

$$\sup_{x \geq n\mathbf{h}} |f(x) - \mathbf{w}_n(x)| \leq \varepsilon \tag{2.5}$$

We put $f_i(x) = \max[f(x, \mathbf{w}_i(x)]$ for $x \geq i\mathbf{h}$. Then, clearly, $f_{i-1}(x) \leq f_i(x)$ for $x \geq i\mathbf{h}$. We denote by $u(x, t)$ the solution of the ith problem with initial condition $u_i(x, 0) = f_i(x)$. Because the function $u_i(x, t)$ is monotonic in x for every fixed t in its domain of definition, then $u_{i-1}(i\mathbf{h}, t) \leq u_i(i\mathbf{h}, t)$ and by the comparison theorem $u_{i-1}(x, t) \leq u_i(x, t)$ for $x \geq i\mathbf{h}$.

We shall show by induction that for each i we have the limit

$$\sup_{x \geq i\mathbf{h}} |u_i(x, t) - \mathbf{w}_i(x)| \to 0 \quad \text{as} \quad t \to \infty \tag{2.6}$$

This is true for $i = n$ because of (2.5) and the stability of the solution with respect to small perturbations. Suppose condition (2.6) is valid for some $i = k$. Because

$$\mathbf{w}_{k-1}(x) \leq u_{k-1}(x, t) \leq u_k(x, t) \leq \mathbf{w}_k(x) + \varepsilon/2, \quad x \geq k\mathbf{h}$$

is valid for sufficiently large t, it follows from (2.4) that the solution $u_{k-1}(x, t)$ lies in an ε-neighbourhood of the steady-state solution $\mathbf{w}_{k-1}(x)$, and the convergence of (2.6) for $i = k - 1$ follows from the stability of $\mathbf{w}_{k-1}$ with respect to small perturbations.

We obtain the required convergence of the original function problem from the $i = 0$ case of (2.6).

If we no longer assume that $f(x)$ is a monotonic function, then when the conditions

$$0 \leq f(x) \leq u^0 \quad \text{for } x \geq 0, \quad f(x) \to 0 \quad \text{as} \quad x \to +\infty \tag{2.7}$$

are satisfied, the convergence of the solution of problem (1.1), (1.2), (2.3) to the steady-state solution $\mathbf{w}(x)$ will follow from the comparison theorems if the initial function can be estimated from above and below by monotonic functions whose convergence has already been proved.

We also note that it follows from the monotonicity of the steady-state solutions and their global stability that a steady-state solution is unique.

Thus we have proved the following fundamental theorem.

Theorem. With the above assumptions about the function $F(u)$ (conditions 1–3 of Sec. 1) there exists a steady-state solution $\mathbf{w}(x)$ of problem (1.1), (1.2). If the matrix $F^*[\mathbf{w}(x)]$ is functionally undecomposible, then this solution is unique in the class of bounded functions, has a negative derivative $\mathbf{w}^*(x)$ for $x \geq 0$ and is stable both with respect to small perturbations and globally (when conditions (2.7) are satisfied). Here stability is to be understood in the uniform norm with respect to perturbations that vanish at $x = 0$ and $x = +\infty$.

3. PARAMETER DEPENDENCE OF THE SOLUTION

We will now consider a more-general statement of the problem, which is parameter dependent, and obtain estimates for changes in the solution due to changes in the parameters. Suppose we are given an initial boundary-value problem on the half-strip $\Omega = \{(x, y) | x \geq 0, -1 \leq y \leq 1\}$:

$$\frac{\partial u}{\partial t} = \mathbf{a}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - c\frac{\partial u}{\partial x} + F(u, x, y) \tag{3.1}$$

$$u|_{(x, y) \in \partial\Omega} = \varphi(x, y), \tag{3.2}$$

$$u(x, y, 0) = f(x, y) \tag{3.3}$$

As above, we shall consider solutions that tend to zero as $x \to \infty$ and admit estimates $|u(x, y, t)| \leq u_0$. Here we do not assume that the source $F(x)$ and velocity c satisfy the conditions formulated in Sec. 1. For simplicity, we will restrict ourselves to a scalar equation, although all the constructions can be easily carried over to systems of equations [satisfying conditions (1.4)], and also to a multi-dimensional situation and a domain with a different geometry. The dependence of the source F on the spatial variables can reflect an inhomogeneous medium or the presence of external influences.

We shall suppose that near $u = 0$ the function F can be represented in the form

$$F(u, x, y) = -\mathbf{p}u + g(x, y) + \mathbf{h}_0(u, x, y)$$
$$|\mathbf{h}_0(u, x, y)| \leq \delta \text{ for } x \geq 0, \quad |y| \leq 1, \quad |u| \leq u_0$$

where $\mathbf{p}$ is some positive number and the function $\mathbf{h}_0(u, x, y)$ contains quadratic terms.

We consider problem (3.1–(3.3) for $g(x, y) = g_i(x, y)$ $(i = 1, 2)$ and denote its solutions by $u_i(x, y, t)$ respectively. We put $z = u_2 - u_1$. Then z is obviously a solution of the problem

$$\frac{\partial z}{\partial t} = \mathbf{a}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) - c\frac{\partial z}{\partial x} - \mathbf{p}z + g_2(x, y) - g_1(x, y) + \mathbf{h}(x, y, t) \tag{3.4}$$

$$z|_{(x, y)\in\partial\Omega} = 0, \quad z|_{t=0} = 0 \tag{3.5}$$
$$(\mathbf{h}(x, y, t) = \mathbf{h}_0(u_2(x, t), x, y) - \mathbf{h}_0(u_1(x, t), x, y))$$

It is clear from the maximum principle that we have the estimate

$$\|z(x, y, t)| \leq (\mathbf{m} + 2\delta)/\mathbf{p}, \quad x \geq 0, \quad |y| \leq 1, \quad t \geq 0 \tag{3.6}$$

where $\mathbf{m}$ is a positive number for which

$$|g_2(x, y) - g_1(x, y)| \leq \mathbf{m}, \quad x \geq 0, \quad |y| \leq 1$$

In particular, for $\delta = 0$ (the linear case) (3.6) gives an estimate for changes in the solution depending on changes in the inhomogeneity $g(x, y)$.

If the boundary conditions in (3.5) are inhomogeneous,

$$|z_{(x, y)\in\partial\Omega} = \mathbf{r}(x, y)$$

and $|\mathbf{r}(x, y)| \leq \mathbf{m}_1$, then again using the maximum principle and inequality (3.6) we obtain

$$|z(x, y, t)| \leq \max\,[(\mathbf{m} + 2\delta)/\mathbf{p}, \mathbf{m}_1]$$
$$x \geq 0, \quad |y| \leq 1, \quad t \geq 0 \tag{3.7}$$

We also obtain estimates of solutions under changes of domain. Alongside problem (3.4), (3.5) we consider a similar problem in the domain Ω_0, contained in Ω, with boundary conditions

$$z|_{(x, y)\in\partial\Omega} = 0$$

Its solution is denoted by $z_0(x, y, t)$. We will estimate the quantity $|z - z_0|$ in Ω_0. To do this we cite yet another estimate for the solution of problem (3.4), (3.5):

$$|z(x, y, t)| \leq \mathbf{b}(1 - y^2), \quad \mathbf{b} = (\mathbf{m} + 2\delta)/(2\alpha)$$

We write

$$s(x, y, t) = z(x, y, t)|_{(x, y)\in\partial\Omega_0}$$

Then $z(x, y, t)$ is a solution of Eq. (3.4) in the domain Ω_0 with boundary and initial conditions

$$z(x, y, t)|_{(x, y)\in\partial\Omega_0} = s(x, y, t), \; z|_{t=0} = 0$$

We obtain, as above, (assuming for simplicity that $\mathbf{h}$ is a specified function of x and t) that

$$|z - z_0| \leq \sup_{t\geq 0, (x, y)\in\partial\Omega_0} |s(x, y, t)| \leq \mathbf{b}(1 - \alpha^2)$$
$$\mathbf{b} = \max\min \rho(\mathbf{P}_0, \mathbf{P})$$

where the maximum is taken over all points $\mathbf{P}_0$ of the boundary of Ω_0, while the minimum is over all points $\mathbf{P}$ of the boundary of Ω.

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STRESSES ON THE SURFACE OF A RIGID NEEDLE IN AN ORTHOTROPIC ELASTIC MEDIUM†

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Using the general solution of the problem of stress concentrations on the surfaces of rigid ellipsoidal inclusions [1], the three-dimensional problem of stresses on the surface of a completely rigid needle in an unbounded elastic orthotropic medium under the action of a uniform external field is solved. By a needle we mean an ellipsoidal inclusion, one dimension of which is large compared with the other two. Explicit formulas are obtained and investigated for stresses along the principal sections of the needle in the orthotropic medium and over the entire surface of the needle in an isotropic medium. The calculations are performed, apart from the singular terms (large, but finite quantities).

1. THE STRESS $\sigma^{\alpha\beta}(\mathbf{n})$ on the surface of a completely rigid ellipsoidal inhomogeneity in an arbitrary anisotropic medium and a uniform external field $\sigma_0^{\alpha\beta}$ has the form

$$\sigma(\mathbf{n})=F(\mathbf{n})\sigma_0,\quad F(\mathbf{n})=D(\mathbf{n})R,\quad D(\mathbf{n})=cK(\mathbf{n}) \tag{1.1}$$

Here $\mathbf{n}=(n_1, n_2, n_3)$ is the limit normal vector to the ellipsoidal surface with semi-axes a_α ($\alpha = 1, 2, 3$) and $F(\mathbf{n})$ is a tensor concentration coefficient. The tensor $K(\mathbf{n})$ does not depend on the geometry of the inhomogeneity, is expressed in terms of the Fourier transform of the Green tensor of the homogeneous medium, and was obtained explicitly in [1] for an orthotropic medium. The tensor of elastic constants c of the

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